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Novel massive ground states of spin chains in a magnetic field*

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Abstract. Novel *massive* quantum states appearing in spin chains under a strong magnetic field are discussed. These states lead to plateaus in magnetization curves. When the systems are axially symmetric and the field is applied parallel to the symmetry-axis, the phenomena are analogous to metal-insulator transitions. Striking features of the plateau phenomena – exactness and rationality – are explained as consequences the commensurability condition to the underlying lattice. The effects of the planar anisotropy are also discussed in detail.

PACS. 75.10.Jm Quantized spin models – 75.60.Ej Magnetization curves, hysteresis, Barkhausen and related effects – 75.30.Gw Magnetic anisotropy

1 Introduction

The last decade has witnessed a remarkable progress in understanding of low-dimensional spin systems. Particularly in one dimension (1D), quantum fluctuations are strongest and even problems, which are rather easy (and sometimes trivial) in their classical version, may become highly nontrivial ones at the quantum level.

For example, the *classical* 1D Heisenberg model (the O(3)-invariant vector-spin model) can be solved for finite temperatures as well as for zero temperature [1] and the spin quantum number S plays only a *quantitative* role. However, its quantum version has not been solved except for the spin-1/2 case [2] and several analytical and numerical arguments have been given to show that the ground state of it is quite different according to whether the spin quantum number S is integer or not [3,4].

The above conjecture due to Haldane sparked the study of one-dimensional disordered singlet ground states (some people call them *spin liquids*). Among them, there is a class of ground states called valence-bond solid (VBS) states [5,6]. They are shown [5] to possess important features of the integer-S antiferromagnetic Heisenberg chain predicted by Haldane [3]. Now it serves as a good starting point in discussing the Haldane-like spin liquids.

Another simple but interesting ground state with an excitation gap was found long ago by Majumdar and Ghosh [7]. They found that the ground state of the antiferromagnetic $S = 1/2$ Heisenberg model with a frustrating next-nearest-neighbor (NNN) interaction (or the $S = 1/2$ 2-leg zigzag spin ladder) is exactly known when the NNN coupling is half the strength of the nearest-neighbor one; the translational symmetry of the original model is broken in the infinite-volume limit and two degenerate ground states appear. In the following, we will see that interesting magnetization processes are observed when an external magnetic field is applied to similar models.

All massive (or, short-ranged) ground states described above are realized when no external field is applied. Then, we may ask whether this kind of gapped ground states is possible even in the presence of the field. Although there exist several exceptional cases, it has been believed that the text-book figure of the magnetization curve of antiferromagnets is correct *qualitatively*. Namely, the value of magnetization increases monotonically once the field has exceeded some critical value (*e.g. spin-flop field*). This seems quite natural since the vector spins are canted along the applied field and they gradually align as the field is increased. Even for the case of the $S = 1$ Heisenberg antiferromagnetic chain, which has quite a non-classical ground state, the magnetization curve [8,9] does not seem, at least qualitatively, to contradict with the above picture. When the magnetization curve is smooth as a function of the field, as is observed in many experiments performed on (quasi-)one-dimensional spin systems or in numerical calculations [10] carried out for the spin-S Heisenberg model, we may expect the finitely magnetized ground states to be gapless.

In higher dimensions, several examples [11] have been known which do not exhibit monotonically increasing magnetization curves. For example, in the spin 1/2 Heisenberg antiferromagnets on a triangular lattice, various magnetization processes appear according to the strength of the 2-body and the 4-body exchange interactions [12]. These plateaus are explained by a vector-spin model.

However, quite recently, the possibility of having non-smooth magnetization curve has been pointed out by

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* Dedicated to J. Zittartz on the occasion of his 60th birthday

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several authors [13–19]. For some region of the parameter space, magnetization curve contains intermediate[1] plateaus. Furthermore, a plateau predicted in [14] for the $S = 1$ dimerized (or, bond-alternating) chain was experimentally observed [20] at half of the saturated magnetization.

At least for models known so far, the values of magnetization (per site) m^z, at which plateaus appear, seem robust against a *small* change in the model parameters, that is, plateaus appear exactly at certain special values. Throughout this paper, the word magnetization denotes the z-component of total spin (S^z_{tot}) divided by the number of spins.

Moreover, all these values are rational. Therefore, the most important problem to be solved would be to explain the above mentioned (i) *exactness* and (ii) *rationality*.

Of course there may be several approaches to solve this problem. In the present article, we stress analogy to the theory of the charge-density waves (CDW) and metal-insulator transitions (see [21,22] for reviews). In our approach, the two important features of the phenomena are explained analogously to those in metal-insulator transitions.

The plan of the present paper is as follows. In Section 2, we take two typical models as examples and explain how the unexpected plateaus appear naturally in certain limits. The analogy to metal-insulator transitions which help us understand these phenomena is discussed.

Another approach to plateau phenomena bosonization is explained in Section 3. This relies on the fact that the low-energy property of many (axially symmetric) one-dimensional systems can be captured by considering the effective Tomonaga-Luttinger Hamiltonian. Using the machinery of bosonization and the analogy to the theory of CDW, we can to some extent understand why the plateaus appear for certain special values of m^z. This approach is, in a sense, complementary to the one presented in Section 2.

The effects of axial-symmetry breaking anisotropies are discussed in Section 4. Here, by the word "axial-symmetry" we mean rotational symmetry with respect to the direction of the applied field. In general, it simplifies the situation a lot; the field only couples to a conserved quantity and its effect is the same as that of chemical potential. Although close analogy to many-particle systems holds in many respects, the spin chains are quite different from particle systems in that axial-symmetry breaking anisotropies are allowed to exist. Hence, it is important to consider the stability of plateaus against such anisotropies also from experimental viewpoint. It is shown that the effects are different depending upon the types of the gap-generating interactions and that the existence of anisotropy can alter the transitions qualitatively.

The main results are summarized in the final section.

[1] Hereafter we use the terminology *plateau* for plateaus appearing for finite values of magnetization.

2 Massive ground states in a magnetic field

In the present section, we consider the physics underlying the plateau phenomena. Throughout this and the next sections, we assume that the systems are rotationally invariant around a given axis.

Among models which are known to have plateaus, we take the following two ones. The first is the spin-S bond-alternating Heisenberg model with a frustrating next-nearest-neighbor (NNN) interaction:

$$\mathcal{H}_{alt} = \sum_{i:chain} [1-(-1)^i\delta]\mathbf{S}_i{\cdot}\mathbf{S}_{i+1} + J_2 \sum_{j:chain} \mathbf{S}_j{\cdot}\mathbf{S}_{j+2} \; . \tag{1}$$

The strength of bond alternation is controlled by the parameter $\delta(\geq 0)$ (the model becomes uniform when $\delta = 0$ and completely decoupled when $\delta = 1$). Such geometrically frustrated materials have been found [23,24]. Although all directions are equivalent, we assume that the external field is parallel to the z-axis purely for the clarity of the argument. That is, the Zeeman term is given by

$$\mathcal{H}_{Zeeman} = -H \sum_j S^z_j \; . \tag{2}$$

The value of magnetization is, of course, given by dividing $\sum_j S^z_j$ by the number of spins.

The other interesting model is the spin-S Heisenberg chain with the single-ion anisotropy:

$$\mathcal{H}_D = J \sum_j \mathbf{S}_j{\cdot}\mathbf{S}_{j+1} + D \sum_j (S^z_j)^2 \; . \tag{3}$$

The direction of the external field is essential here and it is applied in the z-direction. Of course, the D-term makes sense only for $S \geq 1$. For $m^z = 0$ and $S = 1$, the model was extensively studied [25] in the context of the Haldane-gap problem.

We begin with the first model $\mathcal{H}_{alt}$. The zero-field case ($m^z = 0$) was investigated by many authors (for example, see [7,26–32] and references therein). In the following, we consider what happens to this model for $m^z > 0$.

According to a naive (classical vector-spin) picture described above, the magnetization curve is monotonically increasing as a function of the applied field, namely, lowest magnetic excitations carrying the S^z-quantum number $\pm$ 1 are gapless.

However, by a simple (but quantal) argument, we can see that this is not always true for several models. Consider, for example, the first model with $S = 1$ and $J_2 = 0$. The magnetization process for $\delta = 1$ is trivial since the system becomes an assembly of two-spin systems; magnetization per site m^z abruptly increases from zero to $m^z = 1/2$ at $H = 1$ and then it remains constant until the field H exceeds 2, where m^z again jumps up to the saturated value 1. That is, the plateau at $m^z = 1/2$ appears.

Of course, this is not surprising since the magnetization curve for *finite* (2-spin) systems is always step-like.

Nevertheless, even if we include a small inter-dimer coupling

$$J' \equiv \frac{1-\delta}{1+\delta}\ , \tag{4}$$

we can show that the above $m^z = 1/2$-plateau persists as long as $|J'|$ is sufficiently small whereas the vertical parts of the curve have finite slopes. This can be readily generalized to the spin-S case; there are $2S-1$ intermediate plateaus when $m^z = half-integer$ and $\delta \approx 1$.

The physics here becomes clearer by the following interpretation. First, we regard the case $\delta = 1$ as a kind of the *atomic limit* of the Hubbard model. In this limit, the appearance of the plateau phase (= *insulating phase*) is natural as has been described above. Of course, here the increment of $S^z_{2i} + S^z_{2i+1}$ plays a role of a particle. For spin-chain problems, particles can be different according to the value of magnetization.

For clarity of the argument, we focus on the spin-1 case of $\mathcal{H}_{alt}$. The physics will be most easily understood for $\delta \approx 1$ and $J_2 \approx 0$; the exchange interaction on strong bonds $(1+\delta)$ is dominant and the others may be taken into account by the low-order perturbation. To this end, it is convenient to use the 9 states on a strong bond

$$\begin{aligned}&|\text{singlet}\rangle\ , \quad |\text{triplet}(S^z)\rangle\ (S^z = 0, \pm 1), \\ &|\text{quintuplet}(S^z)\rangle\ (S^z = 0, \pm 1, \pm 2)\end{aligned} \tag{5}$$

as a basis. Within a sector with a given value of S^z_{tot}, only the lowest states are important to the magnetization process. For $\delta = 1, J_2 = 0$, such lowest states can be found quite easily; among the above 9 states, only $|singlet\rangle$ (*vacancy*) and $|triplet(1)\rangle$ (*particle*) ($|triplet(1)\rangle$ (vacancy) and $|quintuplet(2)\rangle$ (particle)) for $0 \leq S^z_{tot} \leq L/2$ (for $L/2 \leq S^z_{tot} \leq L$) contribute to the lowest states. Of course, L denotes the number of spins on a lattice and is assumed to be an integer-multiple of four.

Then, we include a small deviation of δ from unity to allow (weak) correlation between strong bonds. We extend the method used in [19]. The spin operators can be rewritten as 9×9-matrices using the 9 states on a strong bond as basis. The interactions on weak $(1-\delta)$ bonds and the NNN exchange (J_2) can be written down in terms of these 9×9-matrices. However, within the lowest-order perturbation it is sufficient to keep only two of them $|\text{singlet}\rangle$ (vacancy) and $|\text{triplet}(1)\rangle$ (particle) ($|\text{triplet}(1)\rangle$ (vacancy) and $|\text{quintuplet}(2)\rangle$ (particle)) for $0 \leq S^z_{tot} \leq L/2$ (for $L/2 \leq S^z_{tot} \leq L$). Regarding the particles as spinless fermions created on the rth strong bond by the operator $c^\dagger_r$, we can write down the the first-order effective Hamiltonian describing the low-energy dynamics of the above restricted Hilbert space:

$$\begin{aligned}\mathcal{H}^{(I)}_{eff} = &\frac{2}{3}\big[2J_2 - (1-\delta)\big]\sum_r \left(c^\dagger_r c_{r+1} + c^\dagger_{r+1}c_r\right) \\ &+ \frac{1}{4}\big[2J_2 + (1-\delta)\big]\sum_r n_r n_{r+1} + (1+\delta)\sum_r n_r\end{aligned} \tag{6}$$

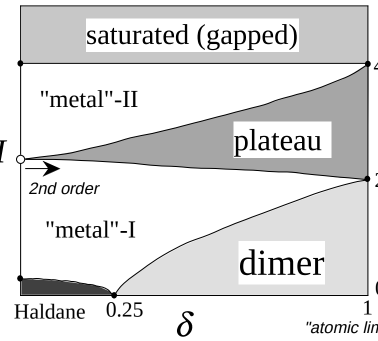

Fig. 1. Schematic δ-H phase diagram for $\mathcal{H}_{alt}$ with $S=1$ and $J_2 = 0$ (see [14] for the precise one). The $m^z = 1/2$ plateau phase is shown as a gray portion. It corresponds to the Mott lobe.

for $0 \leq S^z_{tot} = \sum_{r=1}^{L/2} n_r \leq L/2$, and

$$\begin{aligned}\mathcal{H}^{(II)}_{eff} = &\frac{1}{2}\big[2J_2 - (1-\delta)\big]\sum_r \left\{c^\dagger_r c_{r+1} + c^\dagger_{r+1}c_r\right\} \\ &+ \frac{1}{4}\big[2J_2 + (1-\delta)\big]\sum_r (n_r+1)(n_{r+1}+1) \\ &+ 2(1+\delta)\sum_r n_r + \frac{L}{8}\big[2J_2 + 3\delta + 5\big]\end{aligned} \tag{7}$$

for $L/2 \leq S^z_{tot} = \sum_{r=1}^{L/2}(n_r+1) \leq L$. Note that the presence of the weak bonds and the NNN interactions induce both the hopping ($\sim [2J_2 - (1-\delta)]$) and the nearest-neighbor (density-density) interaction between particles ($\sim [2J_2 + (1-\delta)]$).

The $m^z = 1/2$ plateau mentioned above can be attributed to the change in the "chemical potential" term from $(1+\delta)\sum_j n_j$ (for $0 \leq S^z_{tot} \leq L/2$) to $2(1+\delta)\sum_j n_j$ (for $L/2 \leq S^z_{tot} \leq L$) and exists already in the 2-site limit $J_2 = 0, \delta = 1$. Effective hopping of particles induced by both J_2 and $1-\delta$ reduces the width ($\sim 1+\delta$) of the plateau by an amount $\sim |2J_2 - (1-\delta)|$. Of course, to what extent the above plateau persists is beyond our lowest-order perturbation. However, we may expect that it does down to some critical values of J_2 and δ at which the hopping overcomes the energy cost ($\sim 1+\delta$) on strong bonds. The numerical calculations [14,33] and bosonization argument [16] shows that the critical value $\delta^{lower}_c = 0$, $\delta^{upper}_c = \infty$ for $J_2 = 0$.

If we regard the energy cost as a kind of "Coulomb interactions", this is analogous to what occurs in the metal-insulator transitions [22]; when the "hopping" is much weaker than the "Coulomb interaction", the system is a commensurate (one particle per bond) insulator. The "particles" are almost localized on strong bonds (see Fig. 2b) in the plateau phase. In Figure 1, we show a δ-H phase diagram of $\mathcal{H}_{alt}$ ($S=1$, $J_2 = 0$) schematically (see [14] for the precise one). The region indicated by "plateau" corresponds to the Mott insulating phase in the many-particle language.

However, interestingly enough, the existence of the NNN interaction (J_2) stabilizes other types of

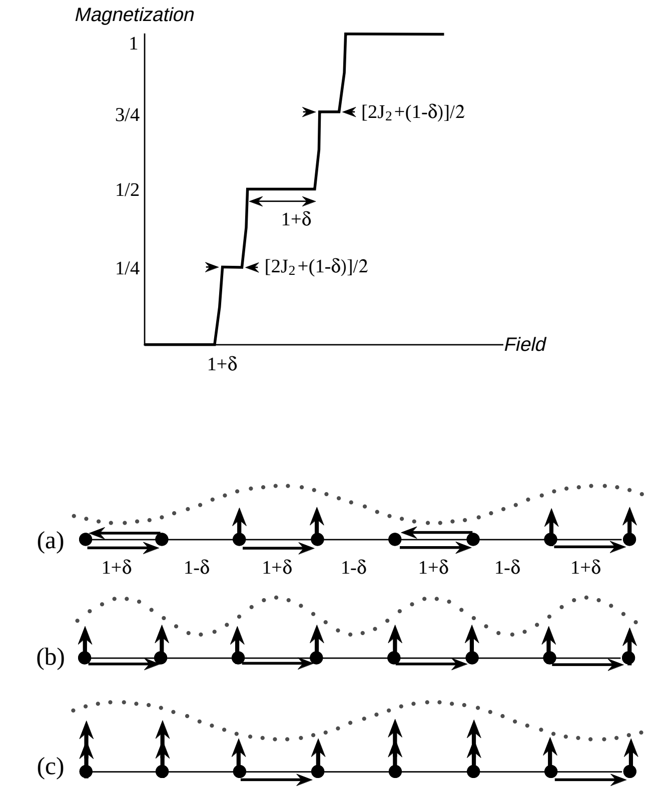

Fig. 2. Schematic drawing of the magnetization curve of the model $\mathcal{H}_{alt}(S=1)$ for $|\delta-1|, J_2 \ll 1$ and expected states at each value of m^z, (a) $m^z = 1/4$, (b) 1/2, and (c) 3/4. Density waves are also shown schematically by dotted lines.

commensurate states at $m^z = 1/4$ and 3/4. To see this, it is convenient to consider the case $2J_2 + \delta = 1$ where the hopping of "particles" vanishes (see (6, 7)). In this case, it is easy to obtain the energy levels of $\mathcal{H}_{eff}^{(I)}$ and $\mathcal{H}_{eff}^{(II)}$; the level spacings change at $m^z = 1/4$, 1/2, and 3/4. This implies that plateaus with width $[2J_2 + (1-\delta)]/2$, $(1+\delta)$, and $[2J_2 + (1-\delta)]/2$ appear at $m^z = 1/4$, 1/2, and 3/4, respectively. Note that the widths of the first- and third plateau are much smaller than that of the second one.

Of course, the hopping ($\sim [2J_2 + \delta - 1]$) again may smear out the plateaus. The critical point can be obtained by noticing that the above effective Hamiltonians are nothing but those of the $S = 1/2$ XXZ chain. From the exact results [34], the plateaus at $m^z = 1/4$ and 3/4 are stable if the Ising anisotropy Δ_{eff} satisfies the condition

$$\Delta_{eff}^{(I)} = \frac{3(2J_2 + (1-\delta))}{16|2J_2 - (1-\delta)|} > 1 \quad \text{for} \quad m^z = \frac{1}{4}$$
$$\Delta_{eff}^{(II)} = \frac{2J_2 + (1-\delta)}{4|2J_2 - (1-\delta)|} > 1 \quad \text{for} \quad m^z = \frac{3}{4} \cdot \qquad (8)$$

Note that the above conditions can not be met for $J_2 = 0$ (a purely alternating chain) or $\delta = 1$ (two-site problem). That is, these plateaus are realized in fan-shaped regions around the Shastry-Sutherland line [7] as consequences of a non-trivial interplay between the bond alternation and the frustrating NNN interaction. Similar mechanism of stabilizing the $m^z = 1/4$-plateau [18] in the $S = 1/2$ case was found in [19]; the region of the plateau for $S = 1/2$ is qualitatively the same as the above ones. We show typical states realized at $m^z = 1/4$, 1/2, and 3/4 in Figures 2a–c.

Now we proceed to the second model $\mathcal{H}_D$. Formally, it resembles the boson Hubbard model discussed in [35]:

$$\mathcal{H}_{boson} = -\frac{1}{2} J \sum_{\langle i,j \rangle} \left(\Phi_i^\dagger \Phi_j + \Phi_j^\dagger \Phi_i \right) + V \sum_j n_j^2 - \mu \sum_j n_j \,. \qquad (9)$$

The similarity would be most clearly seen by recalling the following relations:

$$[n_j, \Phi_k^\dagger] = \delta_{j,k} n_j \quad , \quad [n_j, \Phi_k] = -\delta_{j,k} n_j$$
$$[S_j^z, S_k^\pm] = \pm \delta_{j,k} S_j^\pm \,. \qquad (10)$$

Of course S_j^z may be regarded as a local particle density n_j. When the disorder in the chemical potential μ is absent, it is known [35] that the Mott insulating phase with $\langle n_j \rangle = n$ ($\in \mathbb{Z}$) undergoes the superfluid-insulator transition into the gapless superfluid phase as the ratio J/V is varied.

In the atomic limit ($J = 0$), $\mathcal{H}_D + \mathcal{H}_{Zeeman}$ reduces to the 1-site problem

$$D \sum_j (S_j^z)^2 - H \sum_j S_j^z \qquad (11)$$

and we can easily see that it contains plateaus for $S \geq 3/2$. Again, the nearest-neighbor coupling J competes with the single-ion anisotropy $D(>0)$ which stabilizes the plateaus. Hence, we may expect that the plateau becomes more and more unstable with the ratio D/J decreased and that below a certain critical value they disappears. For $S = 3/2$, the critical value is obtained in [15,36].

However, it differs from the boson Hubbard model in the following respects. First, there is no free kinetic part for $S \geq 1$; the *spin current*

$$\sum_j \frac{-i}{2} J \left(S_{j+1}^+ S_j^- - S_j^+ S_{j+1}^- \right) \qquad (12)$$

satisfying the continuity equation is *not* conserved even for the XY-part. In addition to this, J already contains the "density-density" interaction.

3 Bosonization and CDW-picture

We have presented simple arguments clarifying the physics underlying the plateau phenomena. They are valid in the vicinity of the atomic limit ($J \approx 0$ for $\mathcal{H}_D$ and $\delta \approx 1$ for $\mathcal{H}_{alt}$).

On the other hand, however, there is yet another method appropriate to investigate the weak-coupling limit-bosonization [37]. In the present section, we give an alternative phenomenological approach using bosonization.

In the present section, we consider spin chains (or, quasi-one-dimensional systems) which are rotationally invariant with respect to some symmetry axis (say, z-axis). The external magnetic field is applied along this axis and hence S^z_{tot} is conserved.

In the absence of a magnetic field, spin chains show a variety of low-energy behaviors [3,38,39]. However, when the field is strong enough, they will become gapless; the gapless low-energy degrees of freedom will be carried by the staggered component of the azimuthal angle ϕ of each spin which represents long-wavelength (antiferromagnetic) fluctuations. The soliton number of ϕ is related to the conserved S^z_{tot}.

Such a low-energy Hamiltonian would be given by that of a harmonic liquid, or, the Luttinger liquid named by Haldane [40]. The model contains two important phenomenological parameters "sound velocity" v_S and the compactification radius[2] R. To relate them to physical observables, we consider the twisted boundary condition both along the spatial (chain) direction and along the imaginary-time direction. Namely, we impose the following two boundary conditions on the azimuthal angles in performing the path-integral: (i) $\phi(x+L,\tau)=\phi(x,\tau)+\Phi_0^{(x)}$ ($\tau\in[0,\beta]$), and (ii) $\phi(x,\beta)=\phi(x,0)+\Phi_0^{(\tau)}$. The dimension along the imaginary-time axis (τ-axis) β denotes the inverse temperature $1/(k_BT)$.

The Tomonaga-Luttinger Hamiltonian describing the long-wavelength fluctuations of the azimuthal angle around the antiferromagnetic configuration is given by

$$\mathcal{H}_{TL}=\int_0^L \frac{dx}{2\pi}v_S\left[\frac{\pi^2}{R^2}:\Pi^2:+R^2:(\partial_x\phi)^2:\right], \quad (13)$$

where the angular variable ϕ is defined on a unit circle, *i.e.* $\phi\sim\phi+2\pi$. The canonical momentum Π and the azimuthal angle ϕ satisfy the commutation relation $[\phi(x),\Pi(y)]_{equal\ time}=i\delta(x-y)$. In the following, it is convenient to define the dual field $\widetilde{\phi}$ by

$$\Pi(x)\equiv\frac{1}{\pi}\partial_x\widetilde{\phi}(x)\ . \quad (14)$$

Passing to the path-integral representation of the (low-energy) partition function, we obtain

$$Z(\beta)=\int\mathcal{D}\phi\exp\left\{-\int_0^\beta d\tau\int_0^L dx\frac{1}{2}\left[\left(\frac{R^2}{\pi v_S}\right)(\partial_\tau\phi)^2 + \left(\frac{v_SR^2}{\pi}\right)(\partial_x\phi)^2\right]\right\}\ . \quad (15)$$

In the path-integral formalism, the effect of imposing the type-(i) boundary condition (*spatial twist*) can be realized by a change of variable $\phi(x,\tau)\equiv\phi_{new}(x,\tau)+\Phi_0^{(x)}x/L$ with ϕ_{new} obeying the *periodic* boundary condition $\phi_{new}(x+L)=\phi_{new}(x)$. Plugging this into $\mathcal{H}_{TL}$ and expanding the free energy up to $(\Phi_0^{(x)}/L)^2$, we can easily see that the coefficient of the term $(\partial_x\phi)^2$ is nothing but the spin stiffness D_{spin} and is given by

$$D_{spin}=\left(\frac{v_SR^2}{\pi}\right)\ . \quad (16)$$

In order to obtain the partition function under the twisted boundary condition along the τ-direction, we have only to insert the twist operator $e^{-i\sum(S-S_j^z)\Phi_0^{(\tau)}}$ into the trace[3]:

$$Z_{twisted}=\mathrm{Tr}\ e^{-\beta\mathcal{H}}e^{-i\sum(S-S_j^z)\Phi_0^{(\tau)}}\ . \quad (17)$$

This suggests that the twist in the τ-direction is equivalent to applying the *imaginary* field in the z-direction. Again expanding the free energy in powers of $\Phi_0^{(\tau)}$, we obtain

$$\frac{1}{L}F(\Phi_0)=f_{PBC}+\left(\frac{i\Phi_0^{(\tau)}}{\beta}\right)\times(S-m^z(H))+\frac{1}{2}\left(\frac{\Phi_0^{(\tau)}}{\beta}\right)^2\chi(H;\beta)+\cdots \quad (18)$$

From this, it follows that the susceptibility $\chi(H;\beta)$ is nothing but the "stiffness" along the (imaginary-) time direction, which couples to $(\partial_\tau\phi)^2$ instead of $(\partial_x\phi)^2$. The simple form of $\mathcal{H}_{TL}$ enables us to obtain the following β-independent expression

$$\chi(H)=\left(\frac{R^2}{\pi v_S}\right)\ . \quad (19)$$

Of course, the relations

$$\chi(H)D_{spin}=\left(\frac{R^2}{\pi}\right)^2,\quad \frac{D_{spin}}{\chi(H)}=v_S^2 \quad (20)$$

are well-known in the hydrodynamic theory of magnets. The equation (20) provides a useful way to calculate R (and exponents) directly from physical observables χ and D_{spin}. It is a matter of taste which we take as a fundamental set of parameters (v_S,R) or $(\chi(H),D_{spin})$. In the plateau phase, the magnetic excitation ("particle") is gapped and the correlation $\langle S_i^+S_j^-\rangle$ is short-ranged. Therefore, we may expect that both quantities D_{spin} and $\chi(H)$ vanish.

For a technical reason, hereafter we use the rescaled field

$$\phi_{new}\equiv R\phi \quad (21)$$

instead of ϕ ($\widetilde{\phi}$) defined on a circle of a fixed radius 1 (1/2) and omit the label "new".

The CDW approach starts from the following density-wave expression of the spin operators:

$$s^z\sim m^z+\frac{R}{\pi}\partial_x\widetilde{\phi}+const:\cos\left[2\widetilde{k}_Fx-2R\widetilde{\phi}\right]: \quad (22)$$

$$s^\pm\sim\cos(\pi x):e^{\pm i\frac{\phi}{R}}:+:e^{i\pi x\pm i\frac{\phi}{R}}\cos\left[2\widetilde{k}_Fx-2R\widetilde{\phi}\right]:\ . \quad (23)$$

[2] It determines the radius of the circle on which the ϕ-field is defined.

[3] The appearance of $S-S_j^z$ instead of $-S_j^z$ is related to the requirement of the single-valuedness on the spin coherent states.

The first and the second terms in (22) correspond to the averaged density while the third one expresses the local fluctuation in S^z (or, a density wave). If we treat the system *e.g.* $\mathcal{H}_D + \mathcal{H}_{Zeeman}$ *classically*, a soft mode appears only at the wave vector $q = 0$. This is in contrast to the fact that when $H = 0, D = 0$ there occur two soft modes at $q = 0$ and $q = \pi$ which suggest the existence of two types of low-energy fields [6] one is staggered and the other is uniform. That is, a gapless ($q = 2\widetilde{k}_F$) fluctuation around the above density wave is of the non-classical origin (see [10] for an early discussion about the non-classical soft mode). Note that the second term can also be written as $R\Pi(x)$.

The second expression (23) may be derived by using the polar representation

$$S_j^{\pm} = (-1)^j e^{\pm i\frac{\phi}{R}} \sqrt{(S \mp S_j^z)(S \pm S_j^z + 1)} \, . \qquad (24)$$

These expressions can be derived step by step for some restricted models [16]. Alternatively, we may obtain them *assuming* that the low-energy dynamics of spin chains in a strong field is described by a single-component Bose fluid [40]. The equations (22, 23) play a fundamental role in the following argument.

We first determine the *characteristic wave vector* $\widetilde{k}_F$ by an intuitive argument. Let us recall the physical meaning contained in the expression of s^z. Apparently, the first term denotes the (non-zero) average density of s^z and the second one corresponds to the long-wavelength part of the density fluctuation, which ensures the correct commutation relation between ϕ and S^z. The last one represents local density fluctuations. As is well-known in the theory of CDW [21], the second term can be derived from the density-wave part $\vdots \cos[2\widetilde{k}_F x - 2R\widetilde{\phi}] \vdots$ by introducing the notion of a *local* wave vector $k(x)$. The local wave vector $k(x)$ is defined so that the $2k_F$-density wave may incorporate the local distortion caused by the *phason* $\widetilde{\phi}$:

$$2k(x) \equiv \partial_x \left(2\widetilde{k}_F x - 2R\widetilde{\phi} \right) = 2\widetilde{k}_F - 2R\partial_x \widetilde{\phi}(x) \, , \qquad (25)$$

where we have assumed that $2\widetilde{k}_F$ is given by a certain function $f(m^z)$ of the average magnetization m^z. On the other hand, the shift $\widetilde{k}_F \mapsto k(x)$ is caused by the density fluctuation $m^z \mapsto m^z + \delta m^z(x)$ and hence

$$2k(x) - 2\widetilde{k}_F = f'(m^z)\delta m^z(x) = -2R\partial_x \widetilde{\phi}(x) \, . \qquad (26)$$

Since $s^{\pm} \sim e^{\pm i\phi(x)/R}$ creates changes $s^z \mapsto s^z \pm 1$, the fluctuation $\delta m^z(x)$ should equal to $R\partial_x \widetilde{\phi}/\pi$, that is ,

$$f'(m^z) = -2\pi \, . \qquad (27)$$

Requiring that $2\widetilde{k}_F = f(m^z) = 0$ for the saturated (or fully polarized) state with $m^z = S$, we obtain

$$\widetilde{k}_F = \pi(S - m^z) \, . \qquad (28)$$

Note that it agrees with the one deduced [16] from the composite-spin model [38]. The extension to other models such as ladder-type ones is obvious from the argument given in [16]. It is worth mentioning that the same result can be inferred from the Lieb-Schultz-Mattis argument [15].

Obviously, the translation by n sites in the spatial direction can be realized by shifting the ϕ and $\widetilde{\phi}$ instead of performing $x \mapsto x + n$, because these two operations yield the same result for the density waves. To be concrete, the spatial translation by one site can be realized as the following (simultaneous) discrete symmetry-operations:

$$\phi \mapsto \phi + \pi R \quad , \quad \widetilde{\phi} \mapsto \widetilde{\phi} - \frac{\widetilde{k}_F}{R} \quad \cdot \qquad (29)$$

The first one has a simple physical meaning. If we treat spin systems *classically*, spins are aligned in a staggered manner in the xy-plane: $(S_j^+, S_j^-) = ((-1)^j e^{i\phi_0/R}, (-1)^j e^{-i\phi_0/R})$. Hence, we have to increase the azimuthal angle ϕ_0/R by π to translate the wave-pattern by one site. The second operation for $\widetilde{\phi}$ is magnetization-dependent and less obvious. It is important to note that the wave length of the above non-classical $2\widetilde{k}_F$-(S^z) density wave is given by $\lambda = \pi/k_F$ which corresponds, in the $\widetilde{\phi}$-space, to the period π/R of the $\widetilde{\phi}$-field itself.

Now we proceed to investigating how commensurability restricts the types of allowed interactions. We consider a rather general situation. That is, we allow several interactions with different spatial periods (Q_a). In that case, the spatial period Q_{Ham} of the (full) Hamiltonian is given by the least common multiple of periods Q_a of the constituent interactions. Here a problem arises about how to separate interactions (or, more appropriately, perturbations) from the unperturbed part. Although there is no general guiding principle, we should choose the partial Hamiltonian, which is gapless and is described by a single-component Luttinger liquid for low-energies, as the unperturbed part. For example, one possible (and probably most natural) choice would be to take the uniform nearest-neighbor Heisenberg chain as the unperturbed Hamiltonian for $\mathcal{H}_{alt}$ and $\mathcal{H}_D$. If a large gap (or, a large plateau) always opens in the whole parameter space treated, a weak-coupling treatment like this breaks down.

The interaction may formally be written as

$$\mathcal{V} = \sum_{j \in \Lambda} \sum_a J^{(a)}(j) f_a(\{\mathbf{S}_j\}) \, , \qquad (30)$$

where the index j runs over lattice sites. Here we have introduced coupling constants $J^{(a)}(j)$ for different types (labeled by a) of interactions (say, the nearest-neighbor exchange and the single-ion anisotropy). The functions $f_a(\{\mathbf{S}_j\})$ are polynomials of local spin operators within a finite range around the site-j. For example, we prepare the following two sets of $(J^{(a)}, f_a)$ for the bond-alternating NNN chain $\mathcal{H}_{alt}$:

$$\begin{aligned} &\left(J^{(1)}(j) = (-1)^j \delta, \quad f_1(\{\mathbf{S}_j\}) = \mathbf{S}_j \cdot \mathbf{S}_{j+1} \right), \\ &\left(J^{(2)}(j) = J_2, \qquad f_2(\{\mathbf{S}_j\}) = \mathbf{S}_j \cdot \mathbf{S}_{j+2} \right). \end{aligned} \qquad (31)$$

Since $J^{(a)}(j)$ is a function with a period Q_a, the coupling constants $J^{(i)}(j)$ can be expanded in a Fourier series:

$$J^{(a)}(j) = \sum_{K=0}^{Q_a-1} J_K^{(a)} e^{i\frac{2\pi}{Q_a}Kj} \,. \tag{32}$$

After an appropriate continuum limit is taken, polynomials $f_a(\{\mathbf{S}_j\})$ may be expressed in terms of bosonic variables ϕ and $\widetilde{\phi}$ carrying the low-energy degrees of freedom. Since the system we are treating is axially symmetric, terms like $(s^+)^m(s^-)^n$ $(m \neq n)$ are not allowed; the factors $e^{+i\phi/R}$ and $e^{-i\phi/R}$ always cancel each other and only the $\widetilde{\phi}$-field appears.

Therefore, the full interaction can be written as a sum of terms like

$$e^{i\frac{2\pi}{Q_a}Kx} e^{iN(2\widetilde{k}_F x - 2R\widetilde{\phi})} = e^{i(\frac{2\pi}{Q_a}K + 2N\widetilde{k}_F)x} e^{-2iNR\widetilde{\phi}} \quad (N \in \mathbb{Z}). \tag{33}$$

In this expression, some terms will contain oscillating factors and may be neglected in the low-energy theory. The remaining non-oscillatory terms have wave vectors equal to integer-multiples of the reciprocal lattice vector $G = 2\pi/Q_{Ham}$ and hence should satisfy

$$\frac{2\pi}{Q_{Ham}} \frac{Q_{Ham}}{Q_a} K + 2N\widetilde{k}_F \equiv 0 \qquad (\mathrm{mod}\ 2\pi/Q_{Ham}) \,. \tag{34}$$

These interactions are allowed by a finite lattice period Q_{Ham} and hence we may call them the Umklapp interactions. Since $\widetilde{k}_F = \pi(S - m^z)$ and $Q_{Ham}/Q_a \in \mathbb{Z}$, the above condition reads

$$NQ_{Ham}(S - m^z) \in \mathbb{Z} \,. \tag{35}$$

This is nothing but the selection rule obtained for $Q_{Ham} = 1$ in [16] and for general Q_{Ham} in [15].

The integer N is called the *order of the commensurability* of the locking potential. For example, an $N = 2$ interaction is generated from bond alternation for $\mathcal{H}_{alt}$ ($S = 1/2$ and $m^z = 1/4$) [19]; it leads to two degenerate ground states in a magnetic field. The commensurability condition (35) is *independent* of the strength of the interaction and this explains the reason for the exactness.

Of course, (35) *alone* does not imply the existence of a plateau at that value of m^z; in order for the plateau to appear actually, the locking interaction allowed by the condition (35) should be relevant in the renormalization-group sense. However, *once we have known that the plateau does appear*, we can tell from (35) that its position is unchanged by a small change of the parameters which keeps the reciprocal lattice vector the same.

Before concluding this section, let us find the relevant order of the commensurability N for plateaus discussed in the previous section. For the model $\mathcal{H}_D$, the atomic-limit argument tells us that all the density waves appearing in the intermediate plateaus have the same period as that of the underlying lattice. Therefore, the plateaus of the model $\mathcal{H}_D$ are described by the case $N = 1$ in the weak-coupling region. A similar argument applies to the $m^z = 1/2$-plateau of $\mathcal{H}_{alt}(S = 1)$ (see Fig. 2b).

On the other hand, the pattern of the density wave occurring at $m^z = 1/4, 3/4$ for the model $\mathcal{H}_{alt}(S = 1)$ does not exactly match that of the lattice as can be seen in Figure 2a and c; the wave length of the former is twice as large as the period of the lattice and the weak-coupling theory is given by the $N = 2$ model. Of course, all these conclusions derived by an intuitive argument agree with those of the commensurability condition (35).

4 Effects of anisotropy

4.1 Plateau versus XY-Order

In the previous section, we have treated the case of axially symmetric chains where the external field is applied along the symmetry axis (z-axis). In the language of many-particle systems, this is equivalent to introducing the chemical potential coupled to the total particle number. In this case, the situation is rather simple; smooth increase of magnetization implies that magnetic excitations which change S^z_{tot} by $\pm$ 1 are gapless for that value of the field. Although conservation of the particle-number is fundamental in many-particle systems, corresponding conservation law ($[\mathcal{H}, S^z_{tot}] = 0$) is guaranteed only for restricted models in spin systems; in many cases, a certain kind of axial-symmetry-breaking anisotropies exists and hence the above conservation law is violated.

For this reason, we consider the effect of weak anisotropies on the plateaus. For concreteness, we treat (i) the so-called E-term and (ii) the exchange anisotropy in the xy-plane, both of which break the conservation law $[\mathcal{H}, S^z_{tot}] = 0$. The E-term is given by

$$E\sum_j \left[(S^x_j)^2 - (S^y_j)^2\right] = \frac{E}{2}\sum_j \left[(S^+_j)^2 + (S^-_j)^2\right] \tag{36}$$

and it is known to exist in magnetic materials such as NENP.

A term like this is invariant not only under the one-site translation but also under the symmetry operation (the inversion of S^z)

$$S^{\pm}_j \mapsto S^{\mp}_j \quad , \quad S^z_j \mapsto -S^z_j \,, \tag{37}$$

which is equivalent to

$$\phi \mapsto -\phi \quad , \quad \widetilde{\phi} \mapsto \frac{\pi}{2R} - \widetilde{\phi} \,. \tag{38}$$

The same argument applies to the exchange anisotropy

$$\gamma \sum_j (S^x_j S^x_{j+1} - S^y_j S^y_{j+1}) = \gamma \sum_j \frac{1}{2}(S^+_j S^+_{j+1} + S^-_{j+1} S^-_j) \,. \tag{39}$$

Hence we can incorporate the effect of these anisotropies into the continuum theory by adding

$$\int dx : \cos\left(\frac{2\phi}{R}\right) : . \tag{40}$$

Of course, we can construct it explicitly from the continuum expression like (22, 23). Note that it is allowed to exist regardless of the value of m^z (or the commensurability condition).

Thus, we may expect that the low-energy behavior in the vicinity of a plateau (at m^z) can be captured by the following multi-coupling sine-Gordon model

$$\mathcal{H} = \mathcal{H}_{TL} + \lambda_1 \int dx : \cos\left(2NR_B\widetilde{\phi}_B\right) : + \lambda_2 \int dx : \cos\left(\frac{2}{R_B}\phi_B\right) : - \frac{R_B}{\pi} H \int dx \partial_x \widetilde{\phi}_B . \tag{41}$$

The Tomonaga-Luttinger Hamiltonian $\mathcal{H}_{TL}$ is given by (13) with ϕ replaced by ϕ/R and the coupling constant λ_2 is proportional to E or γ. In writing down the above effective action, we have assumed that the locking potential $: \cos 2NR_B\widetilde{\phi}_B :$ responsible for the formation of a plateau is unique for the value of m^z considered. Of course, the external field is supposed to be so strong that the value of magnetization m^z and the integer N satisfy the condition (35). We have written the radius R for the bosonic model as R_B to distinguish it from the one for the fermionized model. It is important to note that the filling (or, $\widetilde{k}_F$) affects only the second term (λ_1). The external field H is added to control the commensurability and is measured from the center of the plateau. That is, the field H in (41) actually denotes $H - H_{center}$.

As can be easily seen, the λ_1 tends to order the system into the density-wave (or, plateau) phase, whereas the λ_2 favors the staggered order in the xy-plane. Therefore, the Hamiltonian (41) models the competition between the plateau formation and the XY-order. This kind of models was studied in the context of commensurate-incommensurate (C-IC) transitions in the presence of dislocations [41][4].

Naive power-counting argument tells us that the phase diagram will differ qualitatively according to the order of the commensurability N. For $N \geq 3$, two cosine-interactions can not be relevant at the same time. In other words, they do not compete with each other for any R_B; for $R_B < \sqrt{2}/N$, a commensurate density-wave phase appears, while a staggered planar phase is realized for $R_B > 1/\sqrt{2}$.

On the other hand, the case $N = 1$ is slightly complicated because the tendency towards the density-wave (λ_1) and that towards the planar order (λ_2) compete in the interval $1/\sqrt{2} < R_B < \sqrt{2}$. In this sense, the order $N = 2$ is marginal; λ_1 and λ_2 weakly compete only for $R_B = 1/\sqrt{2}$. Note that the case $N = 1$ is of direct relevance to *e.g.* the $m^z = 1/2$-plateau in the model $\mathcal{H}_{alt}(S = 1)$ and the model $\mathcal{H}_D(S = 3/2)$; the $m^z = 1/4$ plateau of the model $\mathcal{H}_{alt}$ ($S = 1/2$ or $S = 1$; see the Sect. 2) falls under the $N = 2$ case.

To investigate the effect of two mutually competing interactions, we consider the case $H = 0$ (or, more precisely, $H = H_{center}$) first. Since we have no exact result for the case with both λ_1 and λ_2 non-vanishing, we use the renormalization-group (RG) technique. The RG β-function can be obtained by computing the operator-product expansions [42]. The result for $H = 0$ is given by the following set of three coupled equations:

$$\frac{d\lambda_1}{d\ln L} = (2 - N^2R_B^2)\lambda_1 \tag{42}$$

$$\frac{d\lambda_2}{d\ln L} = \left(2 - \frac{M^2}{4R_B^2}\right)\lambda_2 \tag{43}$$

$$\frac{d\ln R_B}{d\ln L} = -\frac{\pi^2}{2}N^2R_B^2(\lambda_1)^2 + \frac{\pi^2}{2}\frac{M^2}{4R_B^2}(\lambda_2)^2 . \tag{44}$$

In the present case, the so-called spin-wave index M is equal to 2. The case $M = 1$ is important in the C-IC transitions (and also the case of staggered field). It has an obvious line of fixed points (the Gaussian fixed line) $\lambda_1 = \lambda_2 = 0$, $R_B = arbitrary$. Moreover, for $N = 2$, two non-trivial fixed lines appear: (i) $\lambda_1 = \lambda_2, R_B = 1/\sqrt{2}$ and (ii) $\lambda_1 = -\lambda_2, R_B = 1/\sqrt{2}$. The three fixed lines are analogous to those of the $S = 1/2$ XYZ model; they are all belong to the Gaussian universality class with continuously varying exponents.

For $N = 1$, the flow always runs away from the Gaussian fixed line unless either λ_1 or λ_2 is vanishing. When R_B is much smaller than $1/\sqrt{2}$ and the anisotropy λ_2 is sufficiently small, the system is in a commensurate (plateau) phase ($R \searrow 0$, $|\lambda_1| \nearrow \infty$), while an xy-ordered phase is realized when $R_B \gg \sqrt{2}$. The situation is slightly complicated in the interval $1/\sqrt{2} < R_B < \sqrt{2}$ because of the competition between the locking potential and the planar anisotropy.

A simple dimensional argument tells us that the RG-flow is governed by a single dimensionless parameter[5]

$$\frac{(\lambda_1)^{\frac{1}{2-R_B^2}}}{(\lambda_2)^{\frac{1}{2-1/R_B^2}}} \tag{45}$$

and that there is a certain "marginal" surface $\lambda_1 = F(\lambda_2, R_B)$; the density-wave phase is realized only for $\lambda_1 < F(\lambda_2, R_B)$. It is easy to show that $F(\lambda_2, R_B = 1) = \pm\,\lambda_2$. Non-trivial fixed point does not appear within our 1-loop analysis. In other words, two phases are switched across this surface. Shortly by solving a special case explicitly, we argue that despite the result of the 1-loop RG this in fact corresponds to the Ising transition (see Fig. 3 for the schematic λ_1-λ_2-R phase diagram). The appearance of

[4] The presence of U(1)-breaking anisotropy λ_2 corresponds to adding dislocations.

[5] This parameter can be regarded as a ratio between two mass-scales introduced by the locking potential and the anisotropy.

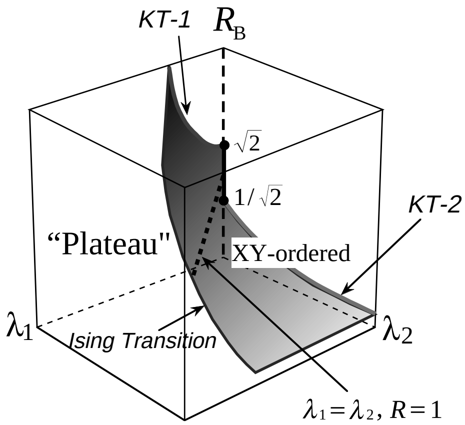

Fig. 3. Sketch of the λ_1-λ_2-R phase diagram obtained from RG-argument and Luther-Emery method. Shaded surface corresponds to the Ising transition, which separates the commensurate plateau phase and the XY-ordered phase.

the Ising critical point is found also in a similar but different (U(1)-symmetric) multi-coupling sine-Gordon model using form-factor approach [43].

For $N = 2$, on the other hand, there appear four critical surfaces on which the system flows onto the non-trivial fixed lines $\lambda_1 = \pm\,\lambda_2$ mentioned above; points off the surfaces flow (suppose that both λ_1 and λ_2 are non-zero) either to the commensurate phase characterized by $R_B \searrow 0$ or to the xy-ordered phase ($R_B \nearrow \infty$). Note that the transitions occurring on them are of the second order[6].

This implies that the existence of any small anisotropy changes the situation not only quantitatively but also *qualitatively*; the gapless ("metallic") phase in the axially symmetric ($\lambda_2 = 0$) case is replaced by the xy-ordered *gapped* one (see the R_B-axis of Fig. 6). Since this gapped excitation is *not* related directly to magnetization, the existence of a gap does not imply the appearance of a plateau in this case. In this sense, the analogy to the Mott insulator is not so useful in the axially asymmetric case.

Furthermore, the presence of the surfaces of second-order transitions replaces a KT phase boundary (open circle in Fig. 6a) separating the gapless phase from the commensurate plateau (or, "Mott-insulating") by a *non-universal* second-order one (open circle in Fig. 6b). Of course, the location of the phase boundary is slightly shifted.

4.2 Transition in a field

Now, we return to the problem of how the magnetization process gets modified by the competition of the two interactions in the $N = 1$ case. The effect of the field H has to be taken account in this subsection. The problem is difficult to solve exactly because the external field H does not couple to a conserved quantity any longer and it plays an essential role. However, some essential features of the $N = 1$ system can be captured by the so-called Luther-Emery technique [45].

For $N = 1$, by tracing back the usual bosonization-mapping, we can rewrite the Hamiltonian (41) as the following fermionic one ($J_{L/R} = \,:\psi^\dagger_{L/R}\psi_{L/R}:\,$):

$$\begin{aligned}\mathcal{H}^{(F)} &= iv_F^{(0)}\int dx\left[-\psi_R^\dagger\partial_x\psi_R + \psi_L^\dagger\partial_x\psi_L\right]\\ &\quad + g_1\int dx : \left(J_L^2 + J_R^2\right) : + g_2\int dx : J_L J_R :\\ &\quad + m\int dx\left[\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L\right]\\ &\quad + g_3\int dx\left[\psi_L^\dagger\psi_R^\dagger + \psi_R\psi_L\right]\\ &\quad - H\int dx\left[:\psi_L^\dagger\psi_L: + :\psi_R^\dagger\psi_R:\right]. \end{aligned} \tag{46}$$

Note that the locking potential (λ_1) and the anisotropy (λ_2) are expressed as the fermion mass-term and the U(1)-breaking term (g_3), respectively. Since we have defined the radius R_F of the fermionic theory by

$$R_F = NR_B\ , \tag{47}$$

the Zeeman term can be rewritten as

$$\begin{aligned}-H\frac{R_B}{\pi}\int dx\partial_x\widetilde{\phi}_B &= -\frac{H}{N}\frac{R_F}{\pi}\int dx\partial_x\widetilde{\phi}_F\\ &= \frac{H}{N}\int dx\left[:\psi_L^\dagger\psi_L: + :\psi_R^\dagger\psi_R:\right].\end{aligned} \tag{48}$$

Similarly, the planar anisotropy is recast as follows:

$$\begin{aligned}&\int dx : \cos\left(\frac{2}{R_B}\phi_B\right)\\ &\to \int dx\left[:(\psi_L^\dagger\psi_R^\dagger)^N: + :(\psi_R\psi_L)^N:\right].\end{aligned} \tag{49}$$

Restricting ourselves to the special case $N = 1$, we obtain (46).

The marginal couplings g_1 and g_2 are chosen so that $\mathcal{H}^{(F)}$ may correctly reproduce the bosonic model (41). For example, we require

$$NR_B = \left[\frac{(v_F^{(0)} + g_1/\pi) - g_2/(2\pi)}{(v_F^{(0)} + g_1/\pi) + g_2/(2\pi)}\right]^{1/4} (= R_F)\ .$$

When $g_2 = 0$, the fermionic Hamiltonian (46) reduces to a free theory and can be diagonalized exactly. The spectra consist of two branches

$$\varepsilon_1(q) = \sqrt{m^2 + g_3^2 + H^2 + (v'_F q)^2 - 2\sqrt{(H^2 + g_3^2)m^2 + H^2(v'_F q)^2}} \tag{50}$$

[6] A similar situation was found [44] for the case $(M, N) = (4, 1)$ in the context of the clock model. What was found in [44] is a second-order transition across the Ashkin-Teller critical line.

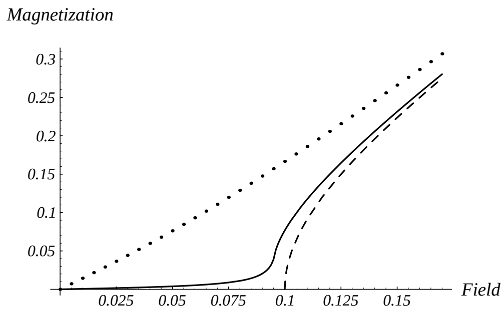

Fig. 4. Magnetization curve of $\mathcal{H}^{(F)}$ for $g_2 = 0$ ($R_F = 1$) and $m = 0.1$. The broken-, solid-, and dotted lines correspond to $g_3 = 0$, 0.03, and 0.12($> m$), respectively. Note that the square-root behavior for $g_3 = 0$ is smeared when g_3 is switched on. The $z = 1$ Ising transition occurs at the critical field $H_c = 0.0954\ldots$ on the solid line.

and

$$\varepsilon_2(q) = \sqrt{m^2 + g_3^2 + H^2 + (v'_F q)^2 + 2\sqrt{(H^2 + g_3^2)m^2 + H^2 (v'_F q)^2}}\ , \tag{51}$$

where $v'_F = v_F^{(0)} + g_1/\pi$. The branch ε_1 is always lower-lying than ε_2 and is important to the critical behavior.

First we consider the case with $m > g_3$, namely, a weakly anisotropic chain. The gap of the first branch

$$|m - \sqrt{g_3^2 + H^2}| \tag{52}$$

closes not at $H = m$ but at a *shifted* critical field $H_c = \sqrt{m^2 - g_3^2}$, where a quantum transition driven by the magnetic field (or, the chemical potential) occurs. On top of the shift of the critical field, the universality class of the transition is altered; instead of a critical point with the dynamical exponent $z = 2$ for $g_3 = 0$ [35,46], the $z = 1$ 2D Ising-type one occurs for $g_3 \neq 0$. Above the critical field, the system is ordered in the XY-plane in a staggered manner. This can be understood intuitively by noticing that the external field drives the system from the non-degenerate $N = 1$ commensurate phase ($H < H_c$) into the two-fold degenerate xy-ordered phase.

Since the external field drives the system in the direction of the (Ising-)temperature[7], the magnetic susceptibility, which corresponds to the specific heat in the Ising model, diverges at H_c *logarithmically*

$$\chi(H) \sim \log|H - H_c| \qquad (H \to H_c \pm 0)\ . \tag{53}$$

Recall that near the edges of the plateau it diverges like $\chi(H) \sim 1/\sqrt{|H - H_c|}$ ($H \to H_c + 0$) for the symmetric case $g_3 = 0$. When the axial anisotropy (g_3) exists, the system is gapped on *both* sides of the critical field, whereas the gap exists only below the critical field when $g_3 = 0$.

The magnetization curve near the plateau can be obtained easily. We show it in Figure 4. The magnetization is measured from the value of the plateau. As long as g_3 is smaller than m, magnetization abruptly increases (the slope has a logarithmic singularity) at $H = H_c (\approx m)$ and the remnant of the plateau may be observed (as the Ising transition). It is important to note that magnetization m^z ever increases although the gap opens even above H_c.

At $g_3 = m$, the critical field vanishes ($H_c = 0$) and the Ising transition occurring here should coincide with the $H = 0$ transition. That is, the marginal surface which we have pointed out in the RG analysis done for $H = 0$ (see Fig. 3) is actually the Ising critical surface.

The situation drastically changes for $g_3 > m$. In this case, the critical field $H_c = \sqrt{m^2 - g_3^2}$ does not exist; there is neither a gap-closing point nor singularity in susceptibility. Consequently the plateau is completely smeared out (see the dotted line of Fig. 4).

Up to now, our analysis has completely neglected the marginal interaction g_2. Taking into account the result of the RG analysis, however, we may expect that the presence of it (*i.e.* non-zero g_2) alters the situation only *quantitatively*[8]; there is the (constant-field $H = H_{center}$) Ising transition separating two different strong-coupling limits (i) $R \nearrow \infty, |\lambda_2| \nearrow \infty$ and (ii) $R \searrow 0, |\lambda_1| \nearrow \infty$. In the density-wave phase, the renormalized parameters satisfies $m^{ren} > g_3^{ren}$; the (Ising) critical field still exist and the remnant of the plateau may be observed as an abrupt increase of the magnetization. In the xy-ordered phase, on the other hand, the critical field disappears and the plateau is wiped out. This situation is summarized in Figures 4 and 5b. The Ising transition point on the R_B-axis corresponds to a point on the (Ising) critical surface in Figure 3. Since the renormalized radius R_B^{ren} approaches 0 as $R_B \to 0$ (the classical limit), the "plateau" phase borders the incommensurate phase on the line $R_B = 0$. It is worth mentioning that the value of magnetization is *not* fixed even in the "plateau" phase, contrary to the axially symmetric case.

The Ising transition in a field was pointed out also by Affleck [47] on the basis of the phenomenological Landau-Ginzburg model in the context of the onset of magnetization of the $S = 1$ Haldane-gap antiferromagnet. Here we argue for a more general case that the Ising universality is unchanged even when the interaction g_2 is present and that the transition point continues to the zero-field ($H = H_{center}$) one (see also [48]).

To summarize, for sufficiently small λ_2, the situation is quite different according to the value of R_B. For $R_B \gtrsim \sqrt{2}$, a small anisotropy orders the system into the gapped xy-phase; no field-induced transition occurs and magnetization gradually increases.

In the interval $1/\sqrt{2} < R_B < \sqrt{2}$, both the locking potential and the XY-anisotropy are relevant. Different phases appear according to the ratio between $(\lambda_1)^{1/(2-R_B^2)}$ and $(\lambda_2)^{1/(2-1/R_B^2)}$; when the former is larger than the latter, the plateau is preferred to the XY-order. In this case, there exists a critical field H_c, where the $z = 1$ Ising

[7] In fact, the combination $\sqrt{g_3^2 + H^2}$ controls the transition.

[8] This would be supported by the fact that no Lorentz-invariant marginal operator exists around the Ising fixed point.

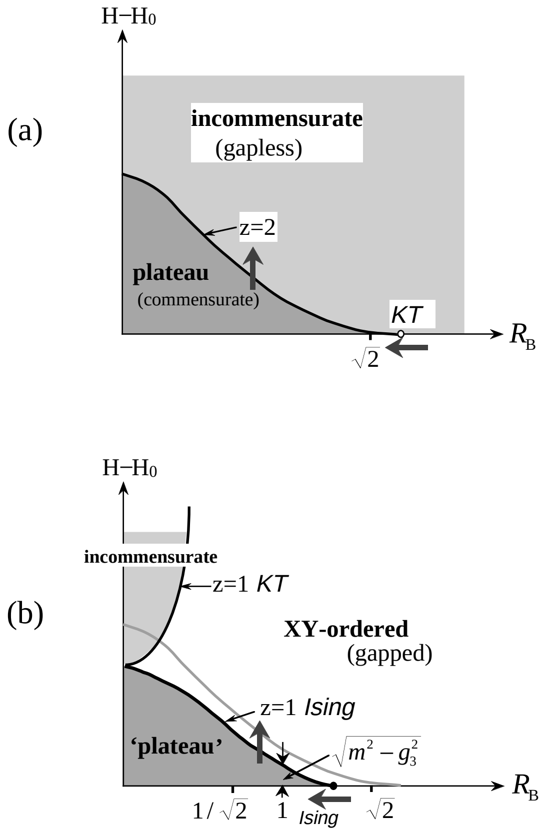

Fig. 5. Schematic R *vs.* field diagram for $N = 1$ in the presence of a weak anisotropy (b). Field H is measured from the center of the plateau The same one for the isotropic case (a) is also shown for comparison. Owing to the competition between two relevant interactions, some of the transitions change their universality class. Note that the word "plateau" is used only to distinguish different phases. Magnetization is not locked strictly even in the "plateau" phase.

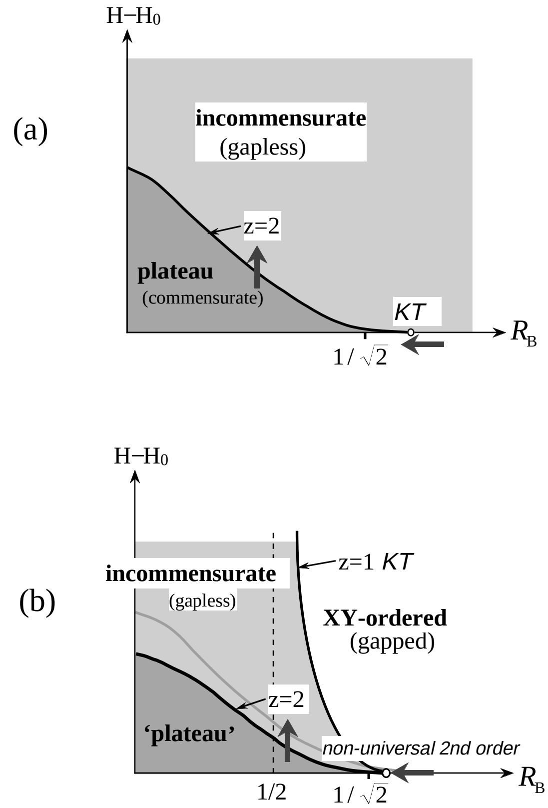

Fig. 6. Schematic R *vs.* field diagram for $N = 2$ in the presence of a weak anisotropy (b). The same one for the isotropic case (a) is also shown for comparison. Transitions are almost unchanged aside from the fact that the infinite-order (KT) incommensurate-plateau transition in the fixed-m^z case is replaced by the second-order XY-plateau one when a small anisotropy is included.

transition occurs with respect to the external field. The existence of two (competing) relevant interactions brings about the Ising transition, which in fact is connected to the same one induced by a field.

For $R_B \lesssim 1/\sqrt{2}$, on the other hand, the plateau phase is stable against the XY-anisotropy. Nevertheless the field-induced transition again belongs to the $z = 1$ Ising universality. This conclusion is derived as follows. First we recall what happens for the case $\lambda_2 = 0$ (or $g_3 = 0$). The relevant λ_2-interaction opens a gap and then it is reduced by an increasing field to vanish at the critical field. According to Schulz [46], the marginal interaction g_2 effectively vanishes at this critical point and hence the system becomes free ($R_F = R_B = 1$). Our result (50) implies that the anisotropy g_3 is *relevant* around this $z = 2$ critical point and that the crossover to another critical point will occur; we may conclude that this is the Ising critical point. That is, the whole phase boundary between the "plateau" phase and the xy-ordered one (see Fig. 5b) including a point on the R_B-axis belongs to the Ising universality.

For $N = 2$, there are no competing interactions; when λ_1 is relevant the XY-anisotropy λ_2 is irrelevant, and *vice versa*. When the radius R_B takes such a small value as to form the plateau, the transition driven by the magnetic field is always of $z = 2$ (note that $z = 1$ can occur as a consequence of the coexistence of two competing relevant interactions for the $N = 1$ case) and thus the remnant of a plateau can be visible as a square-root behavior of magnetization. As R_B (or $|\lambda_2|$) is increased, a second-order transition into the XY-ordered phase occurs, whose exponents depend on initial couplings and hence are non-universal (see Fig. 6b). Recall that such a transition driven by R_B is of the KT-type in the absence of the anisotropy.

If the planar order exists, the magnetization curve is smooth and there is no plateau.

5 Discussion

We have investigated a family of novel gapped ground states realized *in a magnetic field*. In Section 2, we took two models as typical examples to explain how unexpected plateaus appear in the process of magnetization. In the axially symmetric (that is, the systems are invariant under arbitrary rotations with respect to, say, the z-axis) cases, analogy to the Mott insulator is useful in understanding the underlying physics. The lowest excitations having $\delta S^z_{tot} = \pm 1$ are identified with particles and the conserved quantity S^z_{tot} (measured from a certain model-dependent reference point) corresponds to the number of particles; when these particles have a finite gap at some values of magnetization, plateaus are formed there.

At least for the models we have considered, we are able to identify the excitation corresponding to the particles, and effective "Coulomb interactions" and "hoppings" determine whether the particle localizes or not as in the ordinary metal-insulator transitions.

In Section 3, we have presented an alternative approach to the plateau problem. Assuming that the description based on the (single-component) Tomonaga-Luttinger liquid is valid for a wide class of (but not all!) models in a strong field, we can view the occurrence of plateaus as a consequence of the existence of relevant "Umklapp" interactions[9]. They play a role of locking potentials in the CDW; according to their commensurability, different types of commensurate density-wave states will appear, which are identified with the gapped plateau states in a field.

Some of the plateaus which we have considered in the present paper can be generalized to higher dimensions, while bosonization and other methods peculiar to one dimension are no longer applicable. For example, we can easily verify that the model $\mathcal{H}_D$ have plateaus for sufficiently large D (more precisely, sufficiently small J/D). Repeating the mean-field analysis in [35,49], we can show that the critical value $(J/D)_c$ decreases like $1/d$ as the dimensionality d becomes large. Hence we expect that these plateaus are more and more suppressed in higher dimensions.

Then the effects of small axial-symmetry-breaking anisotropies were considered in Section 4. In general, such anisotropies tend to order the system in the xy-plane. In contrast to the symmetric cases, the total magnetization is not associated with any conserved quantities and the analogy to the Mott-insulator is not so useful.

Under several assumptions, we investigated a simplified model the multi-coupling sine-Gordon model by the RG method and the Luther-Emery technique. Because of the presence of interactions with different symmetries, the phase diagrams thus obtained (Figs. 5 and 6) are rich.

In particular, when the order N of the locking potential is unity, the tendencies towards the formation of a plateau and towards the XY-ordering compete with each other. As a result, a new second-order transition occurs. The quantum phase transition induced by the field can change its universality class from the $z = 2$ one, which is the same as the one expected in generic 1-D metal-insulator transitions [35], to the $z = 1$ (Ising) one. Correspondingly, the well-known square-root behavior [50] of magnetization is modified.

Finally, we comment on two complementary pictures for spin systems–*vector picture* and *particle picture*. Of course, the former has a direct analogy to the classical theory of the spin, where a spin is represented as a rigid rotor. Within the vector-picture, magnetization curves might look more or less similar to the classical linear ones. This picture was very successful at least in clarifying the low-energy physics of the Heisenberg-type models in the absence of a field [3,6].

On the other hand, we mainly rely on the particle-picture to explain the exactness and the rationality found in the plateau phenomena. The Umklapp interactions, which come from the discreteness of the underlying lattices and are crucial in explaining the exactness and the rationality, do not exist in the vector-picture. When we chose the path-integral formalism [6], spin systems are formally expressed as interacting classical rigid rotators. At this point, the spins are nothing but unit vectors which are specified by polar and azimuthal angles. In a sufficiently strong field, however, only the slowly-varying component of the azimuthal angles remains in the low-energy sector and the systems look like the Bose fluids [40,51].

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[9] We use the terminology *Umklapp* here in the sense that they are allowed by the finiteness of the lattice period.

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